Introduction to Poisson-Lie Groups

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Tannaka Duality

Our starting point will be the Tannaka duality, which is a dictionary between two worlds, one which we could call algebra, and the other one representation theory or noncommutative geometry. We let k be a field (we care mostly about real or complex numbers). We will be vague enough that what we say can be interpreted in several contexts with more or less technicalities (for instance one could work with topological vector spaces) and we shall ignore issues which come from the fact that infinite dimensional vector spaces are not dualizable.

The basic piece of the dictionary is a correspondence between k-algebras and (k-linear, cocomplete, ...) categories C with a functor (fiber functor) $C \to \mathrm{Vect}_k$. Given an algebra A one can construct the category $C = A-$mod of A-modules. Conversely, given a category C with a fiber functor, one may recover an algebra by taking (the opposite of) the endomorphisms of the fiber functor. One may then think of algebras as just a way of presenting categories.

Now let's add some structure to our categories and see how it reflects on our algebra. We first equip C with a monoidal structure $\otimes$ (compatible with the fiber functor). The corresponding algebraic structure is a bialgebra, an algebra with a compatible comultiplication $(A, \cdot, \Delta)$. If we require our monoidal category to be rigid (every object dualizable) we arrive at the notion of a Hopf algebra, which is a bialgebra together with a morphism (antipode) $S : A \to A$ satisfying appropriate conditions.

Now if we want a braiding on our category, it turns out that one needs to equip the Hopf algebra with an universal R-matrix: this is an invertible element $R \in A \otimes A$ satisfying

$$
\begin{aligned}
(\Delta \otimes 1)R &= R_{13}R_{23} \\
(1 \otimes \Delta)R &= R_{13}R_{12} \\
\Delta^{op} &= R\Delta R^{-1}
\end{aligned}
$$

A Hopf algebra together with an universal R-matrix is called a quasitriangular Hopf algebra. The braiding $V \otimes W = W \otimes V$ is then given by τR, where τ is transposition.

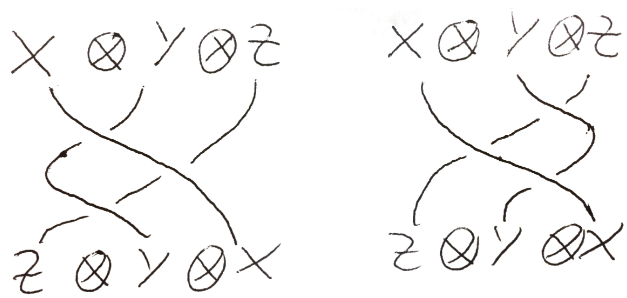
Among the class of quasitriangular Hopf algebras there are two extrema. Triangular Hopf algebras are those for which $R_{21}R = 1 \in A \otimes A$. These give rise to symmetric braided categories. For example, cocommutative Hopf algebras with $R = 1 \otimes 1$ are triangular. These correspond to distributions on Lie groups (i.e, group algebras), and the corresponding category is the category of representations of the group, which is symmetric monoidal in the usual way.

At the other end there are factorizable Hopf algebras, those for which $R_{21}R$ is non degenerate. These are interestingly braided, and are the ones that we really care about.

Why do we care? Historically, the subject grew out of the study of integrable systems. It turns out that to construct these one needs matrices $R \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ satisfying the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in M_n(\mathbb{C})^{\otimes 3}$$

This turns out to be satisfied by the universal R-matrix of a quasitriangular Hopf algebra, after specializing to a representation. The reason this equation holds is due to the fact that there are two ways of applying the braiding axioms to identify $X \otimes Y \otimes Z$ with $Z \otimes Y \otimes X$, corresponding with the following two braids



The two braids are isotopic so the two identifications are the same. Written in terms of R-matrices this yields the Yang-Baxter equation.

A second important application of this is to the construction of Braid/Knot invariants: for any braid one may associate a product of R-matrices as above, which only depends on the topological type of the braid. Invariants of the resulting expression give invariants of the braid. This shows also one reason why one may want factorizable Hopf algebras: one wants the braiding to be interesting to get nontrivial invariants.

Quantization

Now, how does one ever come up with examples of factorizable Hopf algebras? One possible strategy is by quantization. Roughly, this means that one constructs interesting, noncommutative examples of the above structures by deforming commutative ones.

Let's consider the case of algebras first. That means that one starts with a commutative algebra A and wants to deform it to a family A_h of algebras parametrized by $h \in k$, such that $A_0 = A$. Observe that if one has that structure then the commutator on A_h may be written as

$$[\cdot, \cdot] = h\{\cdot, \cdot\} + O(h^2)$$

Therefore if such a deformation exists, we have in particular an operation $\{\cdot, \cdot\} : A \otimes A \to A$ which is both a biderivation and a Lie bracket. This is called a Poisson bracket, and the data of a commutative algebra with a Poisson bracket is called a Poisson algebra. This may be thought of, roughly, as a commutative algebra together with a first order jet of a deformation. The question of reconstructing the deformation A_h from a Poisson algebra is subtle, and is the subject of deformation quantization. One usually calls A_h a quantization of A, and A the (semi)classical limit.

Now, when one works at the classical limit, one sees commutative algebra, and therefore one also sees geometry. We may think of the commutative algebra A as functions on a manifold/variety/scheme, etc M. Then the Poisson bracket, being a biderivation, defines a bivector field $p \in \Gamma(\Lambda^2 TM)$ called the Poisson bivector field. The data of a manifold M together with such a p is called a Poisson manifold. We may think of these as manifolds together with the data of how to deform its algebra of functions.

Of course, one can always set the Poisson bracket to vanish and this will give rise to a constant deformation, which is not so interesting. A more interesting example is the cotangent bundle T^*X to a manifold X. This has a symplectic structure which is the same as a nondegenerate Poisson tensor. A deformation quantization is given by the algebra of quantum observables on X. On local coordinates x^1 this is the algebra generated by symbols x^i, ∂_i with commutation relations $[\partial_i, x^i] = h$. When $h = 1$ one recovers the (filtered) algebra of differential operators on X.

Poisson-Lie Groups

Now let's see how one would construct Hopf algebras from deformation quantization. Hopf algebras have both a multiplication and comultiplication, so there are two (dual) routes we could take: either start with something commutative, or with something cocommutative

1. Let's say we start with $(A, \cdot, \Delta)$ a commutative Hopf algebra. In order to deform it we need the data of a compatible Poisson bracket $\{\cdot, \cdot\}$ on A. We think of A as functions on a manifold, and the comultiplication then defines a product on this manifold. Therefore A may be thought of as functions on a Lie group G. The Poisson bracket then makes G into a Poisson manifold. The data (G, p) of a Lie group with a compatible Poisson structure is called a Poisson-Lie group. These are objects whose algebra of functions can be deformed into a Hopf algebra.

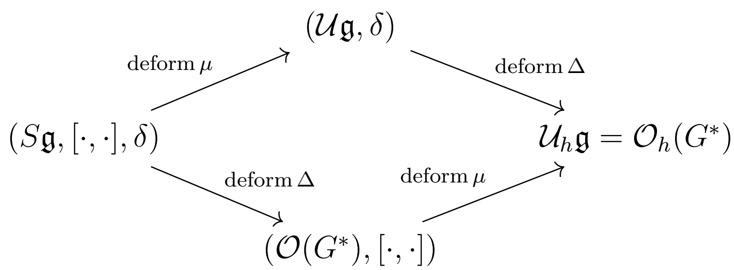
2. We could also start with $(A, \cdot, \Delta)$ a cocommutative Hopf algebra together with a compatible Poisson cobracket $\delta : A \to A \otimes A$. This is dual to the above story, and can be thought of as the Hopf algebra of distributions on a Poisson-Lie group (in other words, the group algebra). One thing usually thought of from this point of view are universal enveloping algebras, which can be thought of as algebras of distributions on a formal group. If one starts with a universal enveloping algebra with a compatible Poisson cobracket, the deformation quantization is called a quantized universal enveloping algebra.

There is a sense in which the above two stories are dual. It turns out that deforming the algebra of distributions on a group is the same as deforming the algebra of functions on a suitable dual group. This duality is made more manifest in the following definition:

Definition. *A Lie bialgebra is a vector space $\mathfrak{g}$ together with a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ and a Lie cobracket $\delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ which is a 1-cocycle for $(\mathfrak{g}, [\cdot, \cdot])$ with values in $\mathfrak{g} \otimes \mathfrak{g}$.*

In other words, this is a pair of dual vector spaces with compatible Lie algebra structures. The Lie algebra of a Poisson-Lie group turns out to be a Lie bialgebra with cobracket the derivative of p.

Now we can put the above two stories on an equal footing. If one has a Lie bialgebra $\mathfrak{g}$, the Hopf algebra $S\mathfrak{g}$ is commutative and cocommutative, and has both a Poisson bracket and Poisson cobracket, so it may be deformed in two different directions (multiplication and comultiplication)



Here G^* is a (formal) Poisson-Lie group with Lie algebra $\mathfrak{g}^*$. The class of Hopf algebras obtained from this procedure sometimes go by the name of quantum groups. Why this name? It turns out that Hopf algebras can be defined internal to any symmetric monoidal category (in which case they are usually called Hopf monoids). When the category is furthermore cartesian, Hopf monoids coincide with group objects. One may think of cartesian categories as classical, while non cartesian ones exhibit some characteristics of linearity/quantum behavior (notably the lack of diagonal maps, which reflects the no-cloning theorem in quantum mechanics). In this sense Hopf algebras form a generalization of the classical notion of groups to a quantum context.

Recall that we were interested in Hopf algebras with a universal R-matrix making its category of modules a braided category. Observe that $\mathcal{U}\mathfrak{g}$ is cocommutative, so $1 \otimes 1 \in \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$ makes it into a triangular Hopf algebra. We would like to deform this into an R-matrix for $\mathcal{U}_h\mathfrak{g}$. This would have a first order expansion

$$R_h = 1 \otimes 1 + hr + O(h)$$

It turns out that r has to belong to $\mathfrak{g} \otimes \mathfrak{g}$, and satisfy

$$\begin{aligned} \delta(X) &= Xr \\ [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] &= 0 \end{aligned}$$

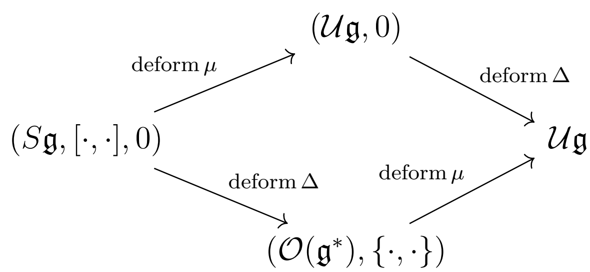
The second equation is called the classical Yang-Baxter equation and arises as the limit of the usual (quantum) Yang-Baxter equation. A Lie bialgebra together with an element r satisfying the above two conditions is called a quasitriangular Lie bialgebra, and r is called its classical R-matrix. Observe that in this context the cobracket is completely determined by the R-matrix.

Among the class of quasitriangular Lie bialgebras we may also identify two opposite cases: triangular Lie bialgebras are those for which $r_{21} + r = 0$; factorizable Lie bialgebras are those for which $r_{21} + r$ is nondegenerate. These correspond to the classical limits of triangular and factorizable Hopf algebras.

There is a classification due to Belavin and Drinfeld of factorizable Lie bialgebra structures on complex simple Lie algebras. If we fix a Cartan subalgebra and choice of positive roots there is a canonical such structure called the standard Lie bialgebra structure. These give rise after quantization to the quantum groups of Drinfeld-Jimbo type.

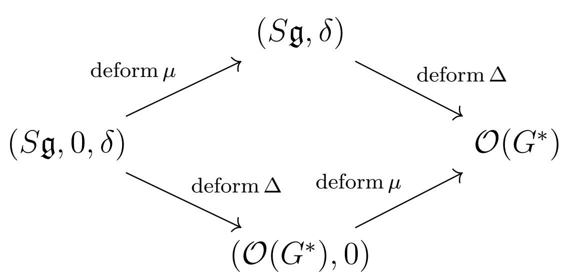
We will not attempt to describe this structure here and will instead content ourselves with understanding the cases were either the bracket or the cobracket vanish.

Example: Let $(\mathfrak{g}, [\cdot,\cdot], 0)$ be a Lie bialgebra with trivial cobracket. This is a triangular Lie bialgebra with $r = 0$. The above diagram of quantizations reads



Here the group G^* is the abelian Lie group $\mathfrak{g}^*$, and the resulting Poisson structure is the Kirillov-Kostant structure. The resulting braided category is the category of representations of $\mathfrak{g}$ (which is symmetric monoidal as expected).

Example: Let $(\mathfrak{g}, 0, \delta)$ be a Lie bialgebra with trivial bracket. This is not quasitriangular unless $\delta = 0$. The diagram in this case is



Here the group G^* is nontrivial but its Poisson bracket vanishes. The category of modules over $\mathcal{O}(G^*)$ is monoidal. The monoidal structure is convolution, which is not commutative if G^* is not abelian, as expected.

Symplectic Leaves

Recall that our starting point was Tannaka duality, which told us that (quasitriangular) Hopf algebras are whatever one needs to present (braided) rigid monoidal categories. Now we know that one may try to construct Hopf algebras by quantization starting with a Lie bialgebra or a Poisson Lie group. One question remains: how is the representation theory of the quantized algebra reflected at the classical limit? In other words, is there a way of constructing representations of quantum groups by studying the Poisson geometry of Poisson Lie groups?

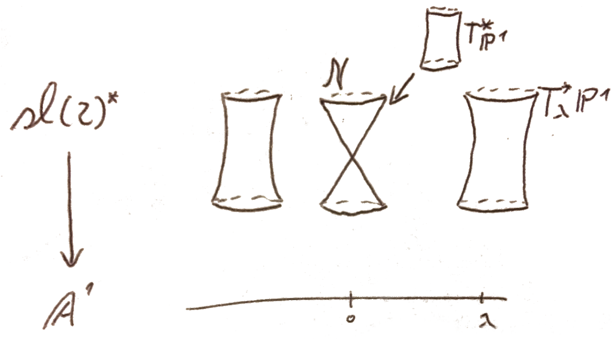
It turns out there is. A first observation is that the quantization A_h of a Poisson algebra $(A, \{\cdot,\cdot\})$ is in general noncommutative (if the Poisson bracket is nontrivial), but it may have nontrivial center. Every time that one has a noncommutative algebra with a nontrivial center, the study of its representation theory breaks down into a part which is the representation theory of its center (which is commutative so this belongs to algebraic geometry) plus the representation theory of something centerless. More precisely, A_h is an algebra over its center $Z(A_h)$, so one may think about it as a family of (centerless!) algebras parametrized by $\mathrm{Spec}(Z(A_h))$. To construct an irreducible representation of A_h one has to fix a point in $\mathrm{Spec}(Z(A_h))$ (i.e, fix a central character) and then give an irreducible representation of the fiber over that point.

There is a similar story in the classical limit for Poisson algebras. A Poisson algebra A contains a Poisson center $Z(A)$ which is the subalgebra of elements which Poisson commute with every other element. Dually, $\mathrm{Spec}(A)$ is fibered over $\mathrm{Spec}(Z(A))$. It turns out that the Poisson tensor is tangent to the fibers and (modulo some subtleties when it drops rank, which we shall ignore) it is nondegenerate when restricted to the fibers. Therefore it corresponds to a symplectic structure on the fibers. It follows then that any Poisson manifold decomposes into a disjoint union of symplectic manifolds, which are called the symplectic leaves. These are the classical limit of the centerless algebras that we had before. Therefore, the representation theory of deformation quantization of Poisson manifolds is obtained by putting together the representation theory of the deformations of its symplectic leaves.

So one reduces the question of constructing representations to the case where the Poisson manifold is symplectic. Modules over the deformation quantization is one incarnation of the A-model of the symplectic manifold. One particular way of getting a representation is by means of geometric quantization. In some cases this will give rise to all (irreducible) representations. We shall illustrate this for the case of Hopf algebras obtained from quantizing Lie bialgebras with trivial bracket or trivial cobracket.

Example: Let $(\mathfrak{g}, [\cdot,\cdot], 0)$ be a Lie bialgebra with trivial cobracket. Its quantization is the universal enveloping algebra $\mathcal{U}\mathfrak{g}$, which can be thought of as a deformation of the algebra of functions on $\mathfrak{g}^*$ in the direction of the Kirillov-Kostant bracket. The symplectic leaves are the coadjoint orbits, and therefore one sees that representations of $\mathfrak{g}$ should arise from geometric quantization of coadjoint orbits. This is called the orbit method.

Let's see how that looks for the case of $\mathfrak{sl}(2,\mathbb{C})$. Call H, E, F the generators of this algebra, with the usual relations $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$. The Poisson center is $\mathbb{C}[c] \subset S\mathfrak{sl}(2)$ where $c = EF + H^2/4$. We therefore get a morphism $\mathfrak{sl}(2)^* \to \mathbb{A}^1$ whose fibers away from 0 are smooth quadrics, and the fiber over 0 is a singular quadric (the nilpotent cone). The symplectic leaves consist of the regular fibers, the nilpotent cone without the origin, and the origin. The nilpotent cone enjoys a resolution of singularities by $T^*\mathbb{P}^1$, called the Springer resolution. The regular fibers are twisted cotangent bundles: they are affine bundles over $\mathbb{P}^1$ modeled over $T^*\mathbb{P}^1$.



One expects therefore by performing geometric quantization that one should be able to find representations of $\mathfrak{sl}(2)$ on global sections of line bundles over $\mathbb{P}^1$. Indeed, the irreducible representations of $\mathfrak{sl}(2)$ consist of $H^0(\mathbb{P}^1, \mathcal{O}(n))$ where n is a nonnegative integer. This is an instance of the Borel-Weil-Bott theorem.

Example: Consider now the case $(\mathfrak{g}, 0, \delta)$ of a Lie bialgebra with trivial bracket. Here the associated Poisson-Lie group is G^* with trivial Poisson tensor. Its symplectic leaves are the points in G^*. We therefore expect a correspondence between points in G^* and irreducible $\mathcal{O}(G^*)$-modules. Indeed, $\mathcal{O}(G^*)$-modules are sheaves on G^*, and one can think of an arbitrary sheaf as a sum of skyscrapper sheaves, which in turn correspond to points in G^*.

References

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[2] S. Majid. *Foundations of quantum group theory.* Cambridge University Press, Cambridge, 1995.